Math 102

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November 20, 2018

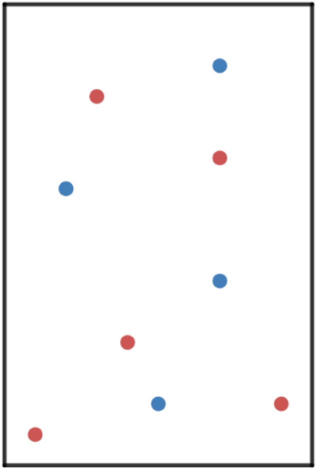
Announcements

- This week's Wednesday office hour is 11:30-1PM (1 hour earlier than normal)

Goals Today

- Logistic Equation - review
  - Law of Mass Action
  - Spread of Disease
- Periodic and Trig functions
  - Sine and Cosine
  - Graphical interpretation
  - Derivatives
  - Period/Frequency, Amplitude, Phase

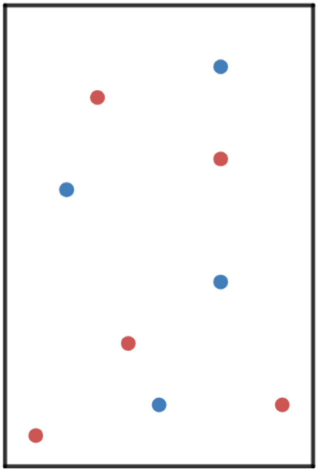
Spread of Disease



$I = \#$ of infected individuals

$S = \#$ of susceptible individuals

Spread of Disease



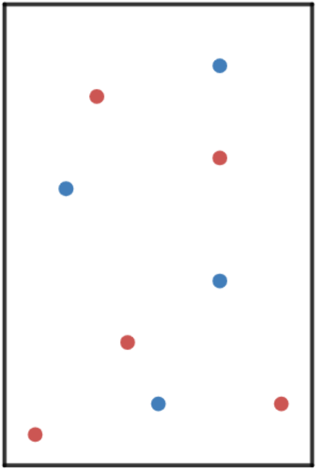
$I = \#$ of infected individuals

$S = \#$ of susceptible individuals

Infection: $s + i \to i + i$

Recovery: $i \to s$

Spread of Disease



$I = \#$ of infected individuals

$S = \#$ of susceptible individuals

Infection: $s + i \to i + i$

Recovery: $i \to s$

The rate of infection is proportional to $S \cdot I$.

The rate of recovery is proportional to I.

Spread of Disease

$$\begin{aligned}\frac{dI}{dt} &= \text{infection} - \text{recovery} \\ &= \beta S I - \mu I\end{aligned}$$

Spread of Disease

$$\frac{dI}{dt} = \text{infection} - \text{recovery}$$
$$= \beta S I - \mu I$$

Question: Let $N = S + I$ be the total population, and assume that N remains constant. Then what is an expression for $\frac{dS}{dt}$?

Spread of Disease

$$\begin{aligned}\frac{dI}{dt} &= \text{infection} - \text{recovery} \\ &= \beta S I - \mu I\end{aligned}$$

Question: Let $N = S + I$ be the total population, and assume that N remains constant. Then what is an expression for $\frac{dS}{dt}$?

$$\frac{dS}{dt} = -\beta S I + \mu I$$

Spread of Disease

$$\frac{dI}{dt} = \beta S I - \mu I$$

Spread of Disease

$$\frac{dI}{dt} = \beta S I - \mu I$$
$$= \beta (N - I) I - \mu I$$

Spread of Disease

$$\begin{aligned}\frac{dI}{dt} &= \beta S I - \mu I \\ &= \beta (N - I) I - \mu I \\ &= -\beta I^2 + (\beta N - \mu) I\end{aligned}$$

Spread of Disease

$$
\begin{aligned}
\frac{dI}{dt} &= \beta S I - \mu I \\
&= \beta(N - I)I - \mu I \\
&= -\beta I^2 + (\beta N - \mu) I \\
&= \beta I \left(-I + N - \frac{\mu}{\beta} \right)
\end{aligned}
$$

Spread of Disease

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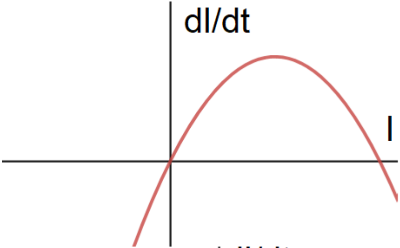
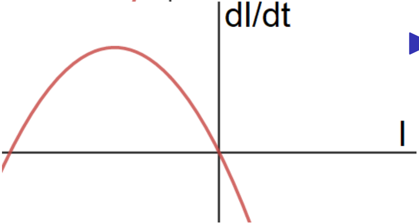
&= \beta(N - I)I - \mu I \\
&= -\beta I^2 + (\beta N - \mu) I \\
&= \beta I \left(-I + N - \frac{\mu}{\beta} \right)
\end{aligned}$$

This is the **logistic equation** with steady states at $I = 0$ and $I = N - \frac{\mu}{\beta}$.

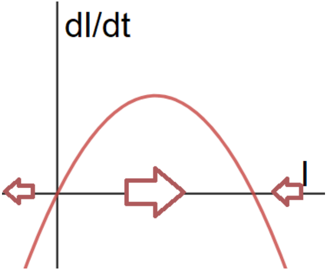
$$\frac{dI}{dt} = \beta I \left(-I + N - \frac{\mu}{\beta}\right)$$



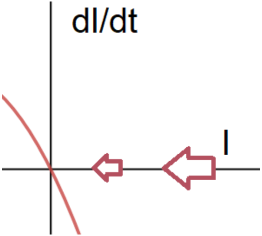


- Question: If $N - \frac{\mu}{\beta} > 0$, then what happens to I over time?
- Question: If $N - \frac{\mu}{\beta} < 0$, then what happens to I over time?

$$\frac{dI}{dt} = \beta I \left(-I + N - \frac{\mu}{\beta} \right)$$



- Question: If $N - \frac{\mu}{\beta} > 0$, then what happens to I over time? $I \to N - \frac{\mu}{\beta}$. The disease becomes endemic.



- Question: If $N - \frac{\mu}{\beta} < 0$, then what happens to I over time? $I \to 0$. The disease disappears.

Define $R_0 = N\beta/\mu$. R_0 is the # of people one person is likely to infect.

- $R_0 > 1 \implies N - \frac{\mu}{\beta} > 0 \implies$ endemic.
- $R_0 < 1 \implies N - \frac{\mu}{\beta} < 0 \implies$ disease disappears.

Define $R_0 = N\beta/\mu$. R_0 is the # of people one person is likely to infect.

- $R_0 > 1 \implies N - \frac{\mu}{\beta} > 0 \implies$ endemic.
- $R_0 < 1 \implies N - \frac{\mu}{\beta} < 0 \implies$ disease disappears.

$$\frac{dI}{dt} = \beta S I - \mu I$$

Question: A disease with a large value of β. Give an interpretation of what this means.

Question: A disease with a small value of μ. Give an interpretation of what this means.

Define $R_0 = N\beta/\mu$. R_0 is the # of people one person is likely to infect.

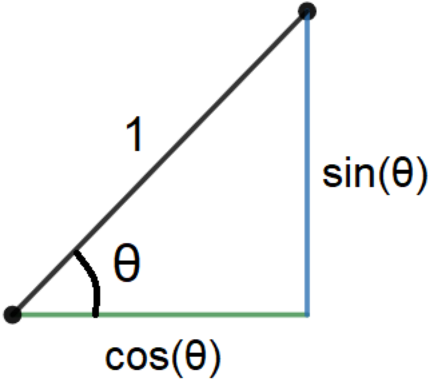
- $R_0 > 1 \implies N - \frac{\mu}{\beta} > 0 \implies$ endemic.
- $R_0 < 1 \implies N - \frac{\mu}{\beta} < 0 \implies$ disease disappears.

$$\frac{dI}{dt} = \beta S I - \mu I$$

Question: A disease with a large value of β. Give an interpretation of what this means. Highly infectious.

Question: A disease with a small value of μ. Give an interpretation of what this means. Slow recovery rate, long incubation period.

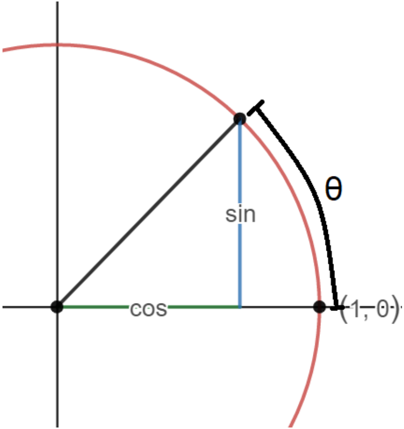
Sine and Cosine - Right-Angled Triangles



$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

Sine and Cosine - Unit Circle

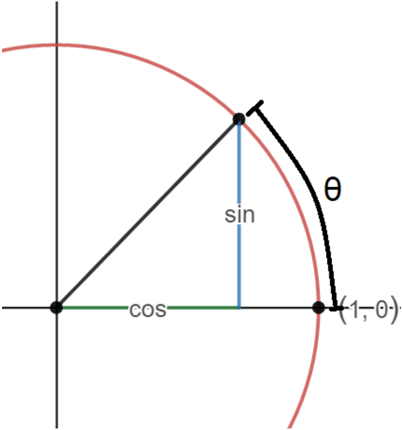


Consider a point that is a distance of θ along the circumference of the circle of radius 1 centered at $(0, 0)$. (starting at $(1, 0)$, going counterclockwise)

$$\sin(\theta) = y - \text{coordinate}$$

$$\cos(\theta) = x - \text{coordinate}$$

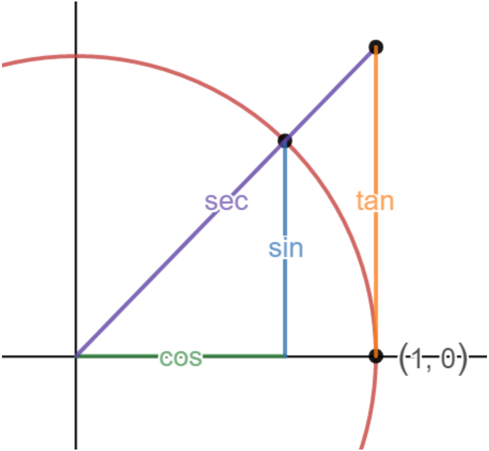
Sine and Cosine - Unit Circle



Remember, the entire circumference of a circle of radius r has length $2\pi r$.

Degrees	Radians
$360°$	2π
$180°$	π
$120°$	$2\pi/3$
$90°$	$\pi/2$
$60°$	$\pi/3$
$\approx 57°$	1
$\vdots$	$\vdots$

Tangent and Secant

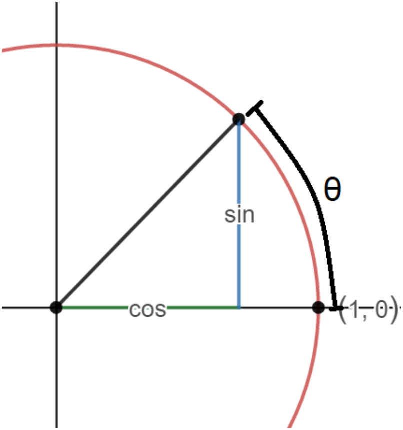


$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

$$\sec(\theta) = \frac{1}{\cos(\theta)}$$

(Every trigonometric function is built out of $\sin(\theta)$ and $\cos(\theta)$.)

Question:



$$\cos(0) =$$

$$\sin(0) =$$

$$\cos(\pi) =$$

$$\sin(\pi) =$$

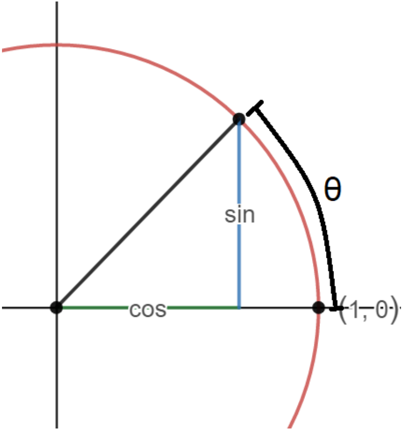
$$\cos(\pi/2) =$$

$$\sin(3\pi/2) =$$

$$\cos(5\pi/2) =$$

$$\sin(-\pi/2) =$$

Question:



$$\cos(0) = 1$$

$$\sin(0) = 0$$

$$\cos(\pi) = -1$$

$$\sin(\pi) = 0$$

$$\cos(\pi/2) = 0$$

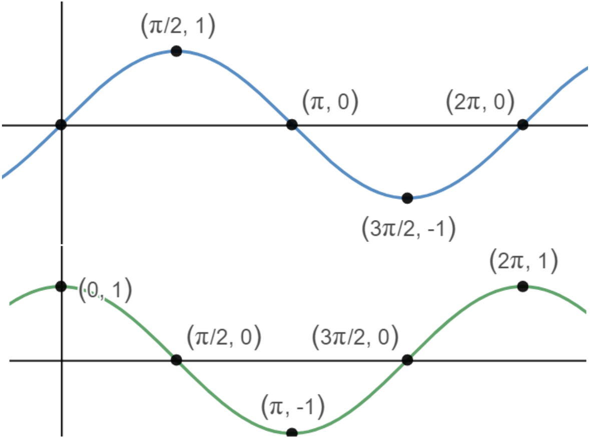
$$\sin(3\pi/2) = -1$$

$$\cos(5\pi/2) = 0$$

$$\sin(-\pi/2) = -1$$

Sine and Cosine - Graphed

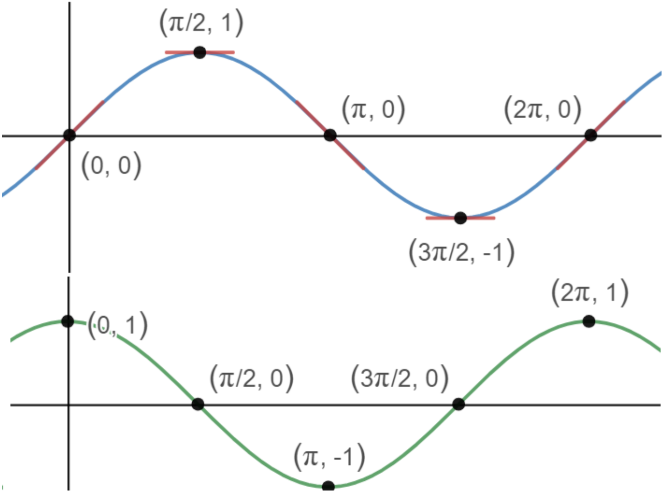
$\sin(t)$ and $\cos(t)$ are **periodic functions**. They have period 2π.



$\sin(t+2\pi) = \sin(t)$

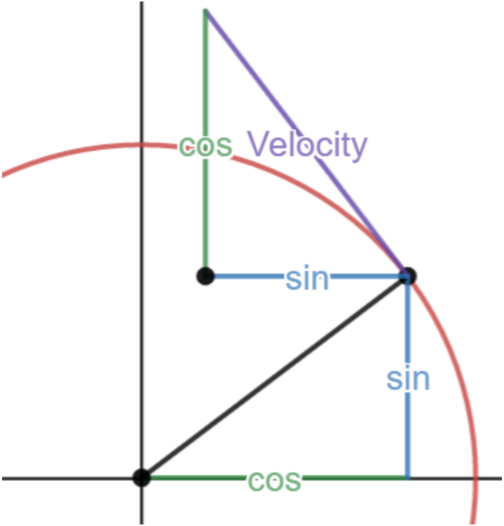
$\cos(t+2\pi) = \cos(t)$

You might also notice that $\cos(t) = \sin(t + \frac{\pi}{2})$!



- $\cos(t) = \sin(t + \frac{\pi}{2})$
- $\cos(t) = (\sin(t))'$ $\qquad -\sin(t) = (\cos(t))'$

$\cos(t) = (\sin(t))'$ and $-\sin(t) = (\cos(t))'$

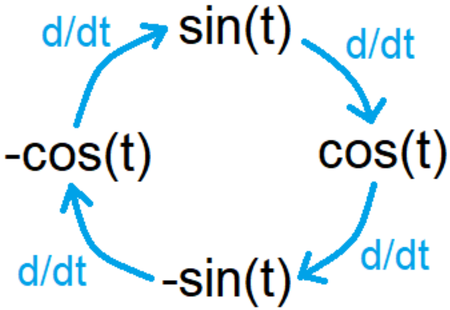


Draw the velocity vector of the point moving around the circle.

$$\text{velocity}_x = -\sin(t)$$

$$\text{velocity}_y = \cos(t)$$

The Circle of Sine



https://www.youtube.com/watch?v=GibiNy4d4gc

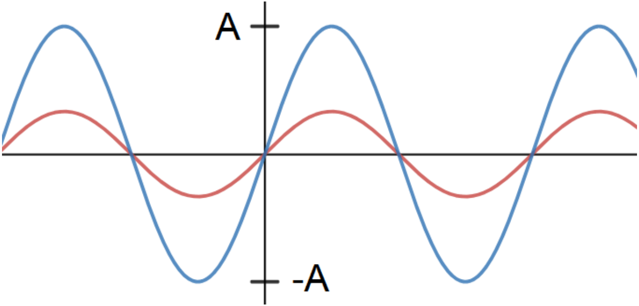
Periodic Functions

A function $f(t)$ is called **periodic** with period P if it satisfies the property

$$f(t + P) = f(t)$$

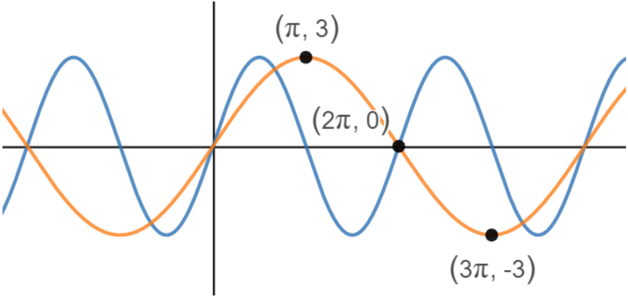
- Molecular vibrations: $P \approx 10^{-14}$ seconds
- Heartbeat: $P \approx 1$ second
- Sunrise/sunset: $P = 1$ day
- Seasons: $P = 1$ year
- Precession of Earth's axis: $P \approx 22{,}000$ years

Amplitude



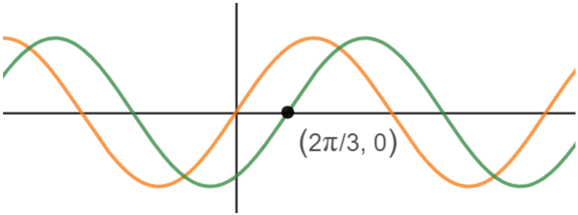
$y = \sin(x)$ $\qquad$ $y = A\sin(x)$

The **amplitude** A of a sinusoidal function is half of the difference between the maximum and minimum values.



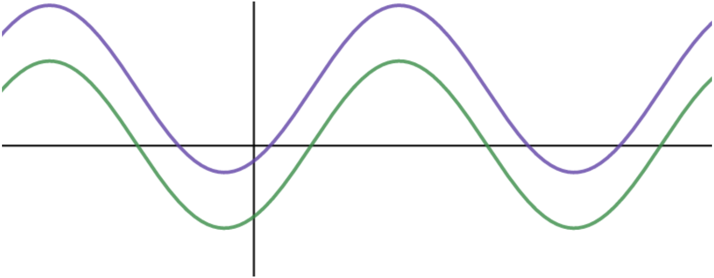
$y = A\sin(x)$ $\quad$ $y = A\sin(\omega x)$

The **frequency** ω of a sinusoidal function is 2π divided by the period.



$y = A\sin(\omega x)$ $y = A\sin(\omega(x - \phi))$

One can introduce a **phase**, which corresponds to a horizontal shift.



$y = A\sin(\omega(x-\phi))$ $\quad$ $y = A\sin(\omega(x-\phi)) + B$

A **sinusoidal function** is a function of the form
$y = A\sin(\omega(x-\phi)) + B$.

Example: Pendulum

https://phet.colorado.edu/sims/html/pendulum-lab/latest/pendulum-lab_en.html

Consider a simple pendulum. The period is given by

$$T = 2\pi\sqrt{L/g}$$

where L is the length, and $g = 9.8\ m/s^2$.

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Question: Let $x(t)$ denote the horizontal position of the pendulum mass.

$$x(t) = A\sin(\omega(t - \phi))$$

If we double the length of the pendulum, what changes?

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Question: Let $x(t)$ denote the horizontal position of the pendulum mass.

$$x(t) = A\sin(\omega(t-\phi))$$

If we double the length of the pendulum, what changes? $A \to 2A$, and $\omega \to \frac{\omega}{\sqrt{2}}$.